

# Stability of two-dimensional Navier-Stokes motions in the periodic case

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**Abstract.** We consider the motion described by the Navier-Stokes equations in a box with periodic boundary conditions. First we prove the existence of global strong two-dimensional solutions. Next we show the existence of global strong three-dimensional solutions under the assumption that the initial data and the external force are sufficiently close to the initial data and the external force of the two-dimensional problem in appropriate spaces. The second result can be treated as stability of strong two-dimensional solutions in the set of suitably strong three-dimensional motions.

MSC 2010: 35Q30, 76D05, 76N10, 35B35, 76D03

Key words: incompressible Navier-Stokes equations, stability of two-dimensional solutions, global regular solutions

## 1. Introduction

The aim of this paper is to prove stability of two-dimensional periodic solutions in the set of three-dimensional periodic solutions to the Navier-Stokes equations. We consider three-dimensional fluid motions in a box  $\Omega = [0, L]^3$ ,  $L > 0$ , described by

$$(1.1) \quad \begin{aligned} v_t + v \cdot \nabla v - \nu \Delta v + \nabla p &= f & \text{in } \Omega \times \mathbb{R}_+, \\ \operatorname{div} v &= 0 & \text{in } \Omega \times \mathbb{R}_+, \\ v|_{t=0} &= v(0) & \text{in } \Omega, \end{aligned}$$

where  $v = (v_1(x, t), v_2(x, t), v_3(x, t)) \in \mathbb{R}^3$  is the velocity of the fluid,  $x = (x_1, x_2, x_3)$  with  $x_i \in (0, L)$ ,  $i = 1, 2, 3$ ,  $p = p(x, t) \in \mathbb{R}$  is the pressure and  $f = (f_1(x, t), f_2(x, t), f_3(x, t)) \in \mathbb{R}^3$  is the external force field.

Finally,  $\nu > 0$  is the constant viscosity coefficient and the dot in the second term of (1.1)<sub>1</sub> denotes the scalar product.

By two-dimensional motions we mean solutions  $(v, p)$  to (1.1) such that  $v = v_s = (v_{s1}(x_1, x_2, t), v_{s2}(x_1, x_2, t), 0) \in \mathbb{R}^2$ ,  $p = p_s(x_1, x_2, t) \in \mathbb{R}$  and  $f = f_s = (f_{s1}(x_1, x_2, t), f_{s2}(x_1, x_2, t), 0) \in \mathbb{R}^2$ .

The main result of this paper is the following. Assume that  $f - f_s$  and  $v(0) - v_s(0)$  are sufficiently small in some norms. Then we show that  $v - v_s$  and  $p - p_s$  are small in appropriate norms for all times. Observe that we are talking about global solutions.

More precisely, two-dimensional periodic solutions satisfy

$$(1.2) \quad \begin{aligned} v_{st} + v_s \cdot \nabla v_s - \nu \Delta v_s + \nabla p_s &= f_s & \text{in } \Omega \times \mathbb{R}_+, \\ \operatorname{div} v_s &= 0 & \text{in } \Omega \times \mathbb{R}_+, \\ v_s|_{t=0} &= v_s(0) & \text{in } \Omega, \end{aligned}$$

where no quantities in (1.2) depend on  $x_3$ . To show stability, we introduce the quantities

$$u = v - v_s, \quad q = p - p_s$$

which are periodic solutions to the problem

$$(1.3) \quad \begin{aligned} u_t + u \cdot \nabla u - \nu \Delta u + \nabla q &= -v_s \cdot \nabla u - u \cdot \nabla v_s + g & \text{in } \Omega \times \mathbb{R}_+, \\ \operatorname{div} u &= 0 & \text{in } \Omega \times \mathbb{R}_+, \\ u|_{t=0} &= u(0) & \text{in } \Omega, \end{aligned}$$

where  $g = f - f_s$ .

Our aim is to show the smallness of  $u(t)$  for all  $t \in \mathbb{R}_+$  if  $u(0)$  and  $g$  are sufficiently small. To derive necessary estimates we use the energy method. Hence the Poincaré inequality is needed. But it does not hold for solutions to problems (1.2) and (1.3). Therefore we introduce the quantities

$$(1.4) \quad \begin{aligned} \bar{v}_s &= v_s - \oint_{\Omega} v_s dx, & \bar{u} &= u - \oint_{\Omega} u dx, & \bar{p}_s &= p_s - \oint_{\Omega} p_s dx, \\ \bar{q} &= q - \oint_{\Omega} q dx, & \bar{f}_s &= f_s - \oint_{\Omega} f_s dx, & \bar{g} &= g - \oint_{\Omega} g dx, \end{aligned}$$

where the integral mean is defined by

$$\oint_{\Omega} \omega dx = \frac{1}{|\Omega|} \int_{\Omega} \omega dx$$

and  $|\Omega| = L^3$ . For the quantities (1.4) problems (1.2) and (1.3) take the form

$$(1.5) \quad \begin{aligned} \bar{v}_{st} + v_s \cdot \nabla \bar{v}_s - \nu \Delta \bar{v}_s + \nabla \bar{p}_s &= \bar{f}_s & \text{in } \Omega \times \mathbb{R}_+, \\ \operatorname{div} \bar{v}_s &= 0 & \text{in } \Omega \times \mathbb{R}_+, \\ \bar{v}_s|_{t=0} &= \bar{v}_s(0) & \text{in } \Omega, \end{aligned}$$

and

$$(1.6) \quad \begin{aligned} \bar{u}_t + u \cdot \nabla \bar{u} - \nu \Delta \bar{u} + \nabla \bar{q} &= -v_s \cdot \nabla \bar{u} - u \cdot \nabla \bar{v}_s + \bar{g} & \text{in } \Omega \times \mathbb{R}_+, \\ \operatorname{div} \bar{u} &= 0 & \text{in } \Omega \times \mathbb{R}_+, \\ \bar{u}|_{t=0} &= \bar{u}(0) & \text{in } \Omega. \end{aligned}$$

For functions  $\bar{v}_s, \bar{u}$  the Poincaré inequality does hold.

Since we are looking for periodic solutions to problem (1.1) we introduce the notation:

$H^m(\Omega) = \{u \in H_{loc}^m(\mathbb{R}^3) : u(x + Le_i) = u(x), i = 1, 2, 3\}$ , where  $e_i, i = 1, 2, 3$  is the canonical basis and

$$H^{2,1}(\Omega \times I) = \{u = u(x, t) : u \in L_2(I; H^2(\Omega)), u_t \in L_2(I; L_2(\Omega))\},$$

where  $I \subset \mathbb{R}$  is an open interval.

Let  $\bar{v}_s(0) \in H^1(\Omega)$ ,  $f_s \in L_{2,loc}(\mathbb{R}_+; H^1(\Omega))$  and denote

$$\bar{A}_1^2(T) = \sup_{k \in \mathbb{N}_0} \int_{kT}^{(k+1)T} \|\bar{f}_s(t)\|_{H^1(\Omega)}^2 dt,$$

where  $T > 0$ ,  $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ ;

$$\begin{aligned} \bar{A}_2^2 &= \|\bar{v}_s(0)\|_{H^1(\Omega)}^2, \\ \bar{A}_3^2(T) &= c_1(\bar{A}_1^2(T) + \bar{A}_2^2) + (\bar{A}_1^2(T) + 1)e^{c_2(\bar{A}_1^2(T) + \bar{A}_2^2)}, \end{aligned}$$

where  $c_1, c_2 > 0$  are some constants;

$$T_* = \frac{2}{c_{s1}} \ln 2,$$

where  $c_{s1} > 0$  is a constant depending on  $\nu$  (introduced in Lemma 2.2).

For a given  $\bar{v}_s(0) \in H^1(\Omega)$  define

$$\mathcal{M} = \{(T, f_s) \in [T_*, \infty) \times L_{2,loc}(\mathbb{R}_+; H^1(\Omega)) : T > \bar{A}_3^2(T)\}.$$

**Definition 1.1.** Let  $(T, f_s) \in \mathcal{M}$ ,  $\bar{v}_s(0) \in H^1(\Omega)$ ,  $\operatorname{div} \bar{v}_s(0) = 0$ . A pair of functions  $(\bar{v}_s, \bar{p}_s)$  is called a strong solution to problem (1.5) if  $\bar{v}_s$  is a weak solution of system (1.5)<sub>1,2</sub> in  $\Omega \times (kT, (k+1)T)$  with the initial condition  $\bar{v}_s|_{t=kT} = \bar{v}_s(kT)$  for all  $k \in \mathbb{N}_0$  and if  $\bar{v}_s \in L_\infty(kT, (k+1)T; H^1(\Omega)) \cap L_2(kT, (k+1)T; H^2(\Omega))$ ,  $\nabla \bar{p}_s \in L_2(\Omega \times (kT, (k+1)T))$  for all  $k \in \mathbb{N}_0$ .

Analogous definition holds for solutions to problem (1.1).

**Theorem 1.1.** Let  $\bar{v}_s(0) \in H^2(\Omega)$ ,  $\operatorname{div} \bar{v}_s(0) = 0$ ,  $f_s \in L_{2,loc}(\mathbb{R}_+; H^1(\Omega))$ ,  $\bar{A}_1^2(T) < \infty$  for all  $T > 0$  and assume that  $(T, f_s) \in \mathcal{M}$ .

Then there exists a unique strong solution  $(\bar{v}_s, \bar{p}_s)$  to problem (1.5) such that

$\bar{v}_s \in H^{2,1}(\Omega \times (kT, (k+1)T)) \cap C([kT, (k+1)T]; H^2(\Omega)) \cap L_2(kT, (k+1)T; H^3(\Omega))$ ,  
 $\nabla \bar{p}_s \in L_2(\Omega \times (kT, (k+1)T))$ ,  $k \in \mathbb{N}_0$  and

$$\|\bar{v}_s\|_{C([kT, (k+1)T]; H^2(\Omega))}^2 + \|\bar{v}_s\|_{L_2(kT, (k+1)T; H^3(\Omega))}^2 \leq c(\bar{A}_1^2, \bar{A}_2^2)(\bar{A}_1^2 + \bar{A}_2^2),$$

where  $c = c(\bar{A}_1^2, \bar{A}_2^2)$  does not depend on  $k$ .

Notice that the set of admissible functions  $f_s$  is large. Let us give two examples of such functions. First define  $f_s = a + h_s$ , where  $a \in \mathbb{R}$  and  $h_s \in L_2(\mathbb{R}_+; H^1(\Omega))$ . Then  $\bar{f}_s = \bar{h}_s \in L_2(\mathbb{R}_+; H^1(\Omega))$  and if  $T > c_1(\int_0^\infty \|\bar{h}_s(t)\|_{H^1(\Omega)}^2 dt + \bar{A}_2^2) + (\int_0^\infty \|\bar{h}_s(t)\|_{H^1(\Omega)}^2 dt + 1)e^{c_2(\int_0^\infty \|\bar{h}_s(t)\|_{H^1(\Omega)}^2 dt + \bar{A}_2^2)} \equiv A_0$  we have  $(T, f_s) \in \mathcal{M}$  for all  $T > \max(T_*, A_0)$ .

The above example shows that there is no restriction on the magnitude of the external force  $f_s$  and for  $a \neq 0$ ,  $f_s$  need not decay in time.

Now, consider another example. Let  $h_s \in L_2(\mathbb{R}_+; H^1(\Omega))$  and  $T > A_0$ . Define a periodic function  $f_{sT}(x, t) = h_s(x, t - kT)$  for  $kT \leq t \leq (k+1)T$ ,  $k \in \mathbb{N}_0$ . Then  $T > \bar{A}_3^2(T)$  and  $(T, f_{sT}) \in \mathcal{M}$  for all  $T > \max(T_*, A_0)$ .

Theorem 1.1 yields the existence of a solution to problem (1.5) such that  $\bar{v}_s \in H^{2,1}(\Omega \times (kT, (k+1)T) \cap C([kT, (k+1)T]; H^2(\Omega)) \cap L_2(kT, (k+1)T; H^3(\Omega)))$ . However, the assumptions of the theorem are too weak to obtain an estimate of  $\|\bar{v}_s\|_{H^{2,1}(\Omega \times (kT, (k+1)T))}$  which is independent of  $k$ . To derive such an estimate we need an additional assumption on  $\bar{f}_s$ , formulated in the theorem below.

**Theorem 1.2.** *Let the assumptions of Theorem 1.1 hold. Moreover, suppose that*

$$\bar{A}_4^2 = \sup_{k \in \mathbb{N}_0} \sup_{kT \leq t \leq (k+1)T} \left| \int_0^t \int_\Omega f_s(x, t') dx dt' + \int_\Omega v_s(0) dx \right|^2 < \infty.$$

*Then there exists a unique strong solution  $(\bar{v}_s, \bar{p}_s)$  to problem (1.5) such that*

$\bar{v}_s \in H^{2,1}(\Omega \times (kT, (k+1)T) \cap C([kT, (k+1)T]; H^2(\Omega)) \cap L_2(kT, (k+1)T; H^3(\Omega)))$ ,  
 $\nabla \bar{p}_s \in L_2(\Omega \times (kT, (k+1)T))$ ,  $k \in \mathbb{N}_0$  and

$$\|\bar{v}_{st}\|_{L_2(kT, (k+1)T; L_2(\Omega))}^2 + \|\bar{v}_s\|_{C([kT, (k+1)T]; H^2(\Omega))}^2 + \|\bar{v}_s\|_{L_2(kT, (k+1)T; H^3(\Omega))}^2 + \|\nabla \bar{p}_s\|_{L_2(\Omega \times (kT, (k+1)T))}^2 \leq c(\bar{A}_1^2, \bar{A}_2^2, \bar{A}_4^2)(\bar{A}_1^2 + \bar{A}_2^2),$$

where  $c = c(\bar{A}_1^2, \bar{A}_2^2, \bar{A}_4^2)$  does not depend on  $k$ .

Using Theorem 1.1 the following theorem concerning the stability of a two-dimensional solution in the set of three-dimensional solutions can be proved. This theorem gives also the existence of a global strong solution to problem (1.1)

**Theorem 1.3.** *Let the assumptions of Theorem 1.1 hold. Let  $v(0) \in H^1(\Omega)$ ,  $\operatorname{div} v(0) = 0$ ,  $f \in L_{2,loc}(\mathbb{R}_+; L_2(\Omega))$  and suppose that*

$$\begin{aligned} \bar{G}(t) = & \sup_{k \in \mathbb{N}_0} \int_{kT}^{(k+1)T} \|\bar{g}(t')\|_{L_2(\Omega)}^2 dt' + \|\bar{u}(0)\|_{L_2(\Omega)}^2 \\ & + \sup_{k \in \mathbb{N}_0} \int_{kT}^{(k+1)T} \left| \int_0^t \int_{\Omega} g dx dt' + \int_{\Omega} u(0) dx \right|^2 dt \\ & + \left| \int_0^t \int_{\Omega} g dx dt' + \int_{\Omega} u(0) dx \right|^2 + \|\bar{g}(t)\|_{L_2(\Omega)}^2 < \infty \quad \text{for all } t \in \mathbb{R}_+. \end{aligned}$$

There exists a constant  $\gamma > 0$  such that if

$$\|u(0)\|_{H^1(\Omega)}^2 \leq \gamma, \quad \bar{G}(t) \leq \varepsilon \gamma \quad \text{for all } t \in \mathbb{R}_+, \quad \text{and some } 0 < \varepsilon < 1,$$

then there exists a unique strong solution  $(v, p)$  to problem (1.1) such that  $v \in H^{2,1}(\Omega \times (kT, (k+1)T))$ ,  $\nabla p \in L_2(\Omega \times (kT, (k+1)T))$ ,  $k \in \mathbb{N}_0$ , and

$$(1.7) \quad \|u(t)\|_{H^1(\Omega)}^2 \leq c\gamma \quad \text{for all } t \in \mathbb{R}_+,$$

where  $c > 0$  is some constant. Moreover,

$$(1.8) \quad \|u\|_{L_2(kT, (k+1)T; H^2(\Omega))}^2 \leq \bar{c}\gamma \quad \text{for all } k \in \mathbb{N}_0,$$

where  $\bar{c} = \bar{c}(T)$ .

Notice that Theorem 1.3 yields the existence of  $v$  in  $H^{2,1}$  while the stability of  $v_s$  in a weaker norm. In the theorem below we formulate the stability result for  $H^{2,1}$ -norm.

**Theorem 1.4.** *Let the assumptions of Theorems 1.2 and 1.3 be satisfied. Moreover suppose that*

$$\sup_{k \in \mathbb{N}_0} \sup_{t \in [kT, (k+1)T]} \left| \int_0^t \int_{\Omega} g(t') dx dt' + \int_{\Omega} u(0) dx \right|^2 \leq \gamma.$$

If  $\gamma$  is sufficiently small then the solution  $(v, p)$  of problem (1.1), which exists in virtue of Theorem 1.3, satisfies

$$(1.9) \quad \|u\|_{H^{2,1}(\Omega \times (kT, (k+1)T))}^2 + \|\nabla q\|_{L_2(kT, (k+1)T; L_2(\Omega))}^2 \leq c\gamma,$$

where  $c = c(T)$ ,  $k \in \mathbb{N}_0$ .

The stability problem for Navier-Stokes equations has been developed in different directions. There are results concerning the stability of weak or regular solutions as well as the stability of two-dimensional solutions or other special solutions in the three-dimensional space. Some papers discuss the question of stability of stationary solutions in the set of nonstationary solutions.

The first results connected with the stability of global regular solutions to the non-stationary Navier-Stokes equations were proved by Beirao da Veiga and Secchi [2], followed by Ponce, Racke, Sideris and Titi [13]. Paper [2] is concerned with the stability in  $L_p$ -norm of a strong three-dimensional solution of the Navier-Stokes system with zero external force in the whole space. In [13], assuming that the external force is zero and a three-dimensional initial function is close to a two-dimensional one in  $H^1(\mathbb{R}^3)$ , the authors showed the existence of a global strong solution in  $\mathbb{R}^3$  which remains close to a two-dimensional strong solution for all times. In [12] Mucha obtained a similar result under weaker assumptions about the smallness of the initial velocity perturbation.

In the class of weak Leray-Hopf solutions the first stability result was obtained by Gallagher [6]. She proved the stability of two-dimensional solutions of the Navier-Stokes equations with periodic boundary conditions under three-dimensional perturbations both in  $L_2$  and  $H^{\frac{1}{2}}$  norms.

The stability of nontrivial periodic regular solutions to the Navier-Stokes equations was studied by Iftimie [8] and by Mucha [10]. The paper [10] is devoted to the case when the external force is a potential belonging to  $L_{r,loc}(\mathbb{T}^3 \times [0, \infty))$  and when the initial data belongs to the space  $W_r^{2-2/r}(\mathbb{T}^3) \cap L_2(\mathbb{T}^3)$ , where  $r \geq 2$  and  $\mathbb{T}$  is a torus. Under the assumption that there exists a global solution with data of regularity mentioned above and assuming that small perturbations of data have the same regularity as above, the author proves that perturbations of the velocity and the gradient of the pressure remain small in the spaces  $W_r^{2,1}(\mathbb{T}^3 \times (k, k+1))$  and  $L_r(\mathbb{T}^3 \times (k, k+1))$ ,  $k \in \mathbb{N}$ , respectively. Paper [8] contains results concerning the stability of two-dimensional regular solutions to the Navier-Stokes system in a three-dimensional torus but here the initial data in the three-dimensional problem belongs to an anisotropic space of functions having different regularity in the first two directions than in the third direction, and the external force vanishes. Moreover, Mucha [11] studies the stability of regular solutions to the nonstationary Navier-Stokes system in  $\mathbb{R}^3$  assuming that they tend in  $W_r^{2,1}$  spaces ( $r \geq 2$ ) to constant flows.

The papers of Auscher, Dubois and Tchamitchian [1] and of Gallagher, Iftimie and Planchon [7] concern the stability of global regular solutions to the Navier-Stokes equations in the whole space  $\mathbb{R}^3$  with zero external force. These authors assume that the norms of the solutions considered decay as  $t \rightarrow \infty$ .

It is worth mentioning the paper of Zhou [14], who proved the asymptotic stability of weak solutions  $u$  with the property:  $u \in L_2(0, \infty, BMO)$  to the Navier-Stokes equations in  $\mathbb{R}^n$ ,  $n \geq 3$ , with force vanishing as  $t \rightarrow \infty$ .

An interesting result was obtained by Karch and Pilarczyk [9], who concentrate on the stability of Landau solutions to the Navier-Stokes system in  $\mathbb{R}^3$ . Assuming that the external force is a singular distribution they prove the asymptotic stability of solution under any  $L_2$ -perturbation.

Paper [5] of Chemin and Gallagher is devoted to the stability of some unique global solution with large data in a very weak sense.

Finally, the stability of Leray-Hopf weak solutions has recently been examined by Bardos et al. [3], where equations with vanishing external force are considered. That paper concerns the following three cases: two-dimensional flows in infinite cylinders under three-dimensional perturbations which are periodic in the vertical direction; helical flows in circular cylinders under general three-dimensional perturbations; and axisymmetric flows under general three-dimensional perturbations. The theorem concerning the first case extends a result obtained by Gallagher [6] for purely periodic boundary conditions. Most of the papers discussed above concern to the case with zero external force ([1]–[3], [5]–[8], [12], [13]) or with force which decays as  $t \rightarrow \infty$  ([18]). Exceptions are [9]–[11], where very special external forces, which are singular distributions in [9] or potentials in [10]–[11], are considered. However, the case of potential forces is easily reduced to the case of zero external forces.

The aim of our paper is to prove the stability result for a large class of external forces  $f_s$  which do not produce solutions decaying as  $t \rightarrow \infty$ . Examples of such functions have been given after the formulation of Theorem 1.1.

It is essential that our stability results are obtained together with the existence of a global strong three-dimensional solution close to a two-dimensional one.

The paper is divided into two main parts. In the first we prove existence of global strong two-dimensional solutions not vanishing as  $t \rightarrow \infty$  because the external force does not vanish either. To prove existence of such solutions we use the step by step method. For this purpose we have to show that the data in the time interval  $[kT, (k+1)T]$ ,  $k \in \mathbb{N}$ , do not increase with  $k$ . For this we also need the time step  $T$  to be sufficiently large. In the second part we prove existence of three-dimensional solutions that remain close to two-dimensional solutions. For this we need the initial velocity and the external force to be sufficiently close in appropriate norms to the initial velocity and the external force of the two-dimensional problems.

The proofs of this paper are based on the energy method, which is available thanks to the periodic boundary conditions. The proofs of global existence which follow from the step by step technique are possible thanks to the natural decay property of the Navier-Stokes equations. This is mainly used in the first part of the paper (Section 3). To prove stability (Section 4) we use smallness of data  $(v(0) - v_s(0))$ ,  $(f - f_s)$  and a contradiction argument applied to the nonlinear ordinary differential inequality (4.11).

We restrict ourselves to proving estimates, because existence follows easily by the Faedo-Galerkin method.

The paper is organized as follows. In Section 2 we introduce notation and give some auxiliary results. Section 3 is devoted to the existence of a two-dimensional solution. It also contains some useful estimates of the solution. In Section 4 we prove the existence of a global strong solution to problem (1.1) close to the two-dimensional solution for all time.

## 2. Notation and auxiliary results

By  $L_p(\Omega)$ ,  $p \in [1, \infty]$ , we denote the Lebesgue space of integrable functions. By  $H^s(\Omega)$ ,  $s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ , we denote the Sobolev space of periodic functions with the finite norm

$$\|u\|_{H^s} \equiv \|u\|_{H^s(\Omega)} = \sum_{|\alpha| \leq s} \left( \int_{\Omega} |D_x^\alpha u|^2 dx \right)^{1/2},$$

where  $D_x^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3}$ ,  $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$ ,  $\alpha_i \in \mathbb{N}_0$ ,  $i = 1, 2, 3$ .

To prove Theorems 1.2, 1.4 we need formulas for the means of  $v_s$  and  $u$ . Hence, we have

**Lemma 2.1.** *Assume that  $\oint_{\Omega} f_s(t) dx, \oint_{\Omega} g(t) dx$  are locally integrable on  $\mathbb{R}_+$  and  $\oint_{\Omega} v_s(0) dx, \oint_{\Omega} u(0) dx$  are finite. Then, for all  $t \in \mathbb{R}_+$ ,*

$$(2.1) \quad \oint_{\Omega} v_s(t) dx = \int_0^t \oint_{\Omega} f_s(t) dx + \oint_{\Omega} v_s(0) dx,$$

$$(2.2) \quad \oint_{\Omega} u(t) dx = \int_0^t \oint_{\Omega} g(t) dx + \oint_{\Omega} u(0) dx.$$

**Proof.** Applying the mean operator to (1.2) and (1.4), integrating by parts and using the periodic boundary conditions, we get

$$(2.3) \quad \frac{d}{dt} \oint_{\Omega} v_s dx = \oint_{\Omega} f_s dx,$$

$$(2.4) \quad \frac{d}{dt} \oint_{\Omega} u dx = \oint_{\Omega} g dx.$$

Integrating (2.3) and (2.4) with respect to time yields (2.1) and (2.2). □

The following lemma follows directly from the Poincaré inequality.

**Lemma 2.2.** *We have*

$$(2.5) \quad c_{s1} \|\bar{v}_s\|_{H^1}^2 \leq \nu \|\nabla \bar{v}_s\|_{L_2}^2,$$

$$(2.6) \quad c_1 \|\bar{u}\|_{H^1}^2 \leq \nu \|\nabla \bar{u}\|_{L_2}^2,$$

where  $c_1, c_{s1}$  are positive constants.



### 3. Two-dimensional solutions

First we need

**Lemma 3.1.** *Assume that*

$$\begin{aligned} 1. \quad & A_1^2 = \frac{1}{c_{s1}} \sup_{k \in \mathbb{N}_0} \int_{kT}^{(k+1)T} \|\bar{f}_s(t')\|_{L_2}^2 dt' < \infty, \\ 2. \quad & A_2^2 = \frac{A_1^2}{1 - e^{-c_{s1}T}} + \|\bar{v}_s(0)\|_{L_2}^2 < \infty, \end{aligned}$$

where  $T > 0$  is fixed and  $c_{s1}$  is introduced in (2.5). Then

$$(3.1) \quad \|\bar{v}_s(kT)\|_{L_2}^2 \leq A_2^2$$

and

$$(3.2) \quad \|\bar{v}_s(t)\|_{L_2}^2 + c_{s1} \int_{kT}^t \|\bar{v}_s(t')\|_{H^1}^2 dt' \leq A_1^2 + A_2^2 \equiv A_3^2$$

for all  $t \in (kT, (k+1)T]$ ,  $k \in \mathbb{N}_0$ .

**Proof.** Multiplying (1.5)<sub>1</sub> by  $\bar{v}_s$ , integrating over  $\Omega$ , using the periodic boundary conditions and inequality (2.5) yields

$$\frac{1}{2} \frac{d}{dt} \|\bar{v}_s\|_{L_2}^2 + c_{s1} \|\bar{v}_s\|_{H^1}^2 \leq \frac{c_{s1}}{2} \|\bar{v}_s\|_{L_2}^2 + \frac{1}{2c_{s1}} \|\bar{f}_s\|_{L_2}^2,$$

where we also applied the Young inequality to the term with the r.h.s. of (1.5)<sub>1</sub>.

Hence, we have

$$(3.3) \quad \frac{d}{dt} \|\bar{v}_s\|_{L_2}^2 + c_{s1} \|\bar{v}_s\|_{H^1}^2 \leq \frac{1}{c_{s1}} \|\bar{f}_{sx}\|_{L_2}^2.$$

Continuing, we obtain

$$\frac{d}{dt} (\|\bar{v}_s\|_{L_2}^2 e^{c_{s1}t}) \leq \frac{1}{c_{s1}} \|\bar{f}_s\|_{L_2}^2 e^{c_{s1}t}.$$

Integrating with respect to time yields

$$\|\bar{v}_s(t)\|_{L_2}^2 \leq \frac{1}{c_{s1}} \int_{kT}^t \|\bar{f}_s(t')\|_{L_2}^2 dt' + e^{-c_{s1}(t-kT)} \|\bar{v}_s(kT)\|_{L_2}^2,$$

for all  $k \in \mathbb{N}_0$ ,  $T > 0$  and  $t \in (kT, (k+1)T]$ . Setting  $t = (k+1)T$  we get

$$\|\bar{v}_s((k+1)T)\|_{L_2}^2 \leq \frac{1}{c_{s1}} \int_{kT}^{(k+1)T} \|\bar{f}_s(t')\|_{L_2}^2 dt' + e^{-c_{s1}T} \|\bar{v}_s(kT)\|_{L_2}^2.$$

By iteration we have

$$\|\bar{v}_s(kT)\|_{L_2}^2 \leq \frac{A_1^2}{1 - e^{-c_{s1}T}} + e^{-c_{s1}kT} \|\bar{v}_s(0)\|_{L_2}^2 \leq A_2^2.$$

Hence, (3.1) is proved. Integrating (3.3) with respect to time from  $t = kT$  to  $t \in (kT, (k+1)T]$ , we obtain (3.2).  $\square$

To obtain an estimate for the second derivatives of  $\bar{v}_s$  we need

**Lemma 3.2.** *Let the assumptions of Lemma 3.1 hold. Let  $\bar{v}_s(0) \in H^1(\Omega)$ ,  $\operatorname{div} \bar{v}_s(0) = 0$ . Suppose that*

$$(3.4) \quad T \geq \frac{2c_{s2}}{c_{s1}} A_3^2,$$

where  $c_{s1}$  is the constant from inequality (2.5),  $c_{s2}$  is introduced in (3.8) below and  $A_3^2$  is defined in Lemma 3.1. Denote

1.  $A_4^2 = c_{s1} e^{c_{s1} A_3^2} A_1^2$ ,
2.  $A_5^2 = \frac{A_4^2}{1 - e^{-\frac{c_{s1}}{2} T}} + \|\bar{v}_{sx}(0)\|_{L_2}^2$ ,
3.  $A_6^2 = A_4^2 + A_5^2$ ,
4.  $A_7^2 = c_{s2}(A_6^2 + 1)A_3^2 + A_5^2$ ,
5.  $A_8^2 = A_3^2 + A_7^2$ .

Then

$$(3.5) \quad \|\bar{v}_{sx}(kT)\|_{L_2}^2 \leq A_5^2$$

and

$$(3.6) \quad \|\bar{v}_{sx}(t)\|_{L_2}^2 + c_{s1} \int_{kT}^t \|\bar{v}_s(t')\|_{H^2}^2 dt' \leq A_8^2$$

for all  $t \in (kT, (k+1)T]$ ,  $k \in \mathbb{N}_0$ .

**Proof.** Differentiating (1.5)<sub>1</sub> with respect to  $x$ , multiplying by  $\bar{v}_{sx}$  and integrating over  $\Omega$  yields

$$\frac{1}{2} \frac{d}{dt} \|\bar{v}_{sx}\|_{L_2}^2 + \nu \|\bar{v}_{sxx}\|_{L_2}^2 \leq \|\bar{v}_{sx}\|_{L_3}^3 + \|\bar{f}_s\|_{L_2} \|\bar{v}_{sxx}\|_{L_2}.$$

Using the Young inequality we get

$$(3.7) \quad \frac{1}{2} \frac{d}{dt} \|\bar{v}_{sx}\|_{L_2}^2 + \frac{\nu}{2} \|\bar{v}_{sxx}\|_{L_2}^2 \leq \|\bar{v}_{sx}\|_{L_3}^3 + \frac{1}{2\nu} \|\bar{f}_s\|_{L_2}^2.$$

Applying the interpolation inequality (see [4])

$$\|u\|_{L_3} \leq c \|u_x\|_{L_2}^{1/3} \|u\|_{L_2}^{2/3}$$

to the first term on the r.h.s. of (3.7), which holds for  $\bar{v}_{sx}$  such that  $\int_{\Omega} \bar{v}_{sx} dx = 0$ , gives

$$(3.8) \quad \frac{d}{dt} \|\bar{v}_{sx}\|_{L_2}^2 + \nu \|\bar{v}_{sxx}\|_{L_2}^2 \leq c_{s2} \|\bar{v}_{sx}\|_{L_2}^4 + c_{s2} \|\bar{f}_s\|_{L_2}^2.$$

In view of inequality (2.5) we have

$$(3.9) \quad \frac{d}{dt} \|\bar{v}_{sx}\|_{L_2}^2 + c_{s1} \|\bar{v}_{sx}\|_{L_2}^2 \leq c_{s2} \|\bar{v}_{sx}\|_{L_2}^4 + c_{s2} \|\bar{f}_s\|_{L_2}^2.$$

Considering inequality (3.9) for  $t \in [kT, (k+1)T]$  implies

$$(3.10) \quad \frac{d}{dt} \left( \|\bar{v}_{sx}\|_{L_2}^2 e^{c_{s1}t - c_{s2} \int_{kT}^t \|\bar{v}_{sx}(t')\|_{L_2}^2 dt'} \right) \leq c_{s2} \|\bar{f}_s\|_{L_2}^2 e^{c_{s1}t - c_{s2} \int_{kT}^t \|\bar{v}_{sx}(t')\|_{L_2}^2 dt'}.$$

Integrating (3.10) with respect to time from  $t = kT$  to  $t \in (kT, (k+1)T]$  we obtain

$$(3.11) \quad \begin{aligned} \|\bar{v}_{sx}(t)\|_{L_2}^2 &\leq e^{c_{s2} \int_{kT}^t \|v_{sx}(t')\|_{L_2}^2 dt'} \cdot \int_{kT}^t \|f_s(t')\|_{L_2}^2 dt' \\ &\quad + e^{-c_{s1}(t-kT)+c_{s2} \int_{kT}^t \|v_{sx}(t')\|_{L_2}^2 dt'} \|\bar{v}_{sx}(kT)\|_{L_2}^2. \end{aligned}$$

Setting  $t = (k+1)T$  in (3.11) and using (3.2) yields

$$(3.12) \quad \begin{aligned} \|\bar{v}_{sx}((k+1)T)\|_{L_2}^2 &\leq e^{c_{s2}A_3^2} \int_{kT}^{(k+1)T} \|f_s(t')\|_{L_2}^2 dt' \\ &\quad + e^{-c_{s1}T+c_{s2}A_3^2} \|\bar{v}_{sx}(kT)\|_{L_2}^2. \end{aligned}$$

In view of assumption (3.4) and notation 1. of the lemma we can write (3.12) briefly as

$$\|\bar{v}_{sx}((k+1)T)\|_{L_2}^2 \leq A_4^2 + e^{-\frac{c_{s1}}{2}T} \|v_{sx}(kT)\|_{L_2}^2.$$

Hence iteration implies (3.5):

$$\begin{aligned} \|\bar{v}_{sx}(kT)\|_{L_2}^2 &\leq \frac{A_4^2}{1 - e^{-\frac{c_{s1}}{2}T}} + e^{-\frac{c_{s1}}{2}kT} \|\bar{v}_{sx}(0)\|_{L_2}^2 \\ &\leq \frac{A_4^2}{1 - e^{-\frac{c_{s1}}{2}T}} + \|\bar{v}_{sx}(0)\|_{L_2}^2 = A_5^2, \end{aligned}$$

where notation 2. is used. Employing (3.5) in (3.11) gives

$$\begin{aligned} \|\bar{v}_{sx}(t)\|_{L_2}^2 &\leq e^{c_{s2}A_3^2} \int_{kT}^{(k+1)T} \|\bar{f}_s(t')\|_{L_2}^2 dt' + e^{-c_{s1}T+c_{s2}A_3^2} A_5^2 \\ &\leq c_{s1}e^{c_{s2}A_3^2} A_1^2 + A_5^2 = A_4^2 + A_5^2 \equiv A_6^2 \end{aligned}$$

for  $t \in [kT, (k+1)T]$ , where we used assumption 1. of Lemma 3.1 together with assumption (3.4) and notation 1. of the present lemma.

Integrating (3.8) with respect to time from  $t = kT$  to  $t \in (kT, (k+1)T]$  we obtain

$$\begin{aligned} \|\bar{v}_{sx}(t)\|_{L_2}^2 + \nu \int_{kT}^t \|\bar{v}_{sxx}(t')\|_{L_2}^2 dt' &\leq c_{s2} \sup_t \|\bar{v}_{sx}(t)\|_{L_2}^2 \\ &\leq c_{s2} \sup_t \|\bar{v}_{sx}(t)\|_{L_2}^2 \int_{kT}^t \|\bar{v}_{sx}(t')\|_{L_2}^2 dt' + c_{s3} \int_{kT}^t \|\bar{f}_s(t')\|_{L_2}^2 dt' + \|\bar{v}_{sx}(kT)\|_{L_2}^2 \\ &\leq c_{s2}[A_6^2 A_3^2 + A_3^2] + A_5^2 \equiv A_7^2. \end{aligned}$$

This implies (3.6) and ends the proof.  $\square$

Inequalities (3.2) and (3.6) imply

$$(3.13) \quad \|\bar{v}_s(t)\|_{H^1}^2 + \int_{kT}^t \|\bar{v}_s(t')\|_{H^2}^2 dt' \leq A_3^2 + A_7^2 \equiv A_8^2$$

for all  $t \in (kT, (k+1)T]$ ,  $k \in \mathbb{N}_0$ .

**Lemma 3.3.** *Suppose there exists a constant  $A_9$  such that*

$$\sup_k \sup_{kT \leq t \leq (k+1)T} \left| \int_0^t \int_{\Omega} f_s(t') dx dt' + \int_{\Omega} v_s(0) dx \right| \leq A_9 < \infty.$$

*Let the assumptions of Lemmas 3.1 and 3.2 hold. Then there exists a solution to problem (1.5) such that  $\bar{v}_s \in H^{2,1}(\Omega \times (kT, (k+1)T))$ ,  $\nabla \bar{p}_s \in L_2(\Omega \times (kT, (k+1)T))$ ,  $k \in \mathbb{N}_0$  and*

$$(3.14) \quad \|\bar{v}_s\|_{H^{2,1}(\Omega \times (kT, (k+1)T))}^2 + \|\nabla \bar{p}_s\|_{L_2(\Omega \times (kT, (k+1)T))}^2 \leq cA_8^2(1 + A_8^2) + cA_8^2A_9^2.$$

**Proof.** Multiplying (1.5)<sub>1</sub> by  $\bar{v}_{st}$ , integrating over  $\Omega$  and with respect to time from  $kT$  to  $(k+1)T$  gives

$$\begin{aligned} \|\bar{v}_{st}\|_{L_2(\Omega \times (kT, (k+1)T))}^2 &\leq c\|\bar{f}_s\|_{L_2(\Omega \times (kT, (k+1)T))}^2 + c \int_{kT}^{(k+1)T} \int_{\Omega} |v_s|^2 |\bar{v}_{sx}|^2 dx dt \\ &\quad + \|\bar{v}_{sx}(kT)\|_{L_2(\Omega)}^2 \leq cA_8^2(1 + A_8^2 + A_9^2), \end{aligned}$$

where

$$\begin{aligned} \int_{kT}^{(k+1)T} \int_{\Omega} |v_s|^2 |\bar{v}_{sx}|^2 dx dt &\leq c\|v_s\|_{L_{\infty}(kT, (k+1)T; H^1(\Omega))}^2 \|\bar{v}_s\|_{L_2(kT, (k+1)T; H^2(\Omega))}^2 \\ &\leq c \left( \|\bar{v}_s\|_{L_{\infty}(kT, (k+1)T; H^1(\Omega))}^2 + \left\| \int_{\Omega} v_s dx \right\|_{L_{\infty}(kT, (k+1)T; H^1(\Omega))}^2 \right) \|\bar{v}_s\|_{L_2(kT, (k+1)T; H^2(\Omega))}^2 \\ &\leq c(A_8^2 + A_9^2)A_8^2. \end{aligned}$$

Next, (1.5)<sub>1</sub> yields

$$\begin{aligned} \|\nabla \bar{p}_s\|_{L_2(\Omega \times (kT, (k+1)T))}^2 &\leq \|\bar{v}_{st}\|_{L_2(\Omega \times (kT, (k+1)T))}^2 \\ &\quad + \|\bar{v}_s\|_{L_2(kT, (k+1)T; H^2(\Omega))}^2 \\ &\quad + c\|v_s\|_{L_{\infty}(kT, (k+1)T; H^1(\Omega))}^2 \|\bar{v}_s\|_{L_2(kT, (k+1)T; H^2(\Omega))}^2 + \|\bar{f}_s\|_{L_2(kT, (k+1)T; L_2(\Omega))}^2 \\ &\leq cA_8^2 + c(A_8^2 + A_9^2)A_8^2. \end{aligned}$$

Hence (3.14) holds. Having estimate (3.14) existence follows by the Faedo-Galerkin method. This concludes the proof.  $\square$

To prove stability of 2d solutions we need more regular 2d solutions than the one given in Lemma 3.2. Namely, we need

**Lemma 3.4.** *Let the assumptions of Lemma 3.2 be satisfied. Suppose that:*

1.  $A_{10}^2 = \sup_k \int_{kT}^{(k+1)T} \|\bar{f}_{sx}(t)\|_{L_2}^2 dt < \infty$ ,
2.  $A_{11}^2 = \exp(c_{s3}A_8^2)c_{s3}A_{10}^2$ ,  $c_{s4} = \frac{c_{s3}}{c_{s1}}$ ,
3.  $T$  is so large that  $-c_{s1}T/2 + c_{s4}A_8^2 \leq 0$ ,
4.  $T$  is so large that  $1 - e^{c_{s1}T/2} \geq 1/2$ ,
5.  $A_{12}^2 = 2A_{11}^2 + \|\bar{v}_{sxx}(0)\|_{L_2}^2 < \infty$ ,
6.  $A_{13}^2 = A_{11}^2 + A_{12}^2 \exp(c_{s4}A_8^2)$ ,
7.  $A_{14}^2 = c_{s3}(A_{13}^2A_8^2 + A_{10}^2) + A_{12}^2$ ,

where  $c_{s3} > 0$  is the constant from (3.19) below. Then

$$(3.15) \quad \|\bar{v}_{sxx}(t)\|_{L_2}^2 \leq A_{13}^2,$$

$$(3.16) \quad \|\bar{v}_{sxx}(t)\|_{L_2}^2 + c_{s1} \int_{kT}^t \|\bar{v}_{sxx}(t')\|_{H^1}^2 dt' \leq A_{14}^2$$

for all  $t \in (kT, (k+1)T]$ ,  $k \in \mathbb{N}_0$ .

**Proof.** Differentiating (1.5)<sub>1</sub> twice with respect to  $x$ , multiplying the result by  $\bar{v}_{sxx}$ , integrating over  $\Omega$  and by parts yield

$$(3.17) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \|\bar{v}_{sxx}\|_{L_2}^2 + \nu \|\nabla \bar{v}_{sxx}\|_{L_2}^2 &= - \int_{\Omega} \bar{v}_{sxx} \cdot \nabla \bar{v}_s \cdot \bar{v}_{sxx} dx \\ &\quad - 2 \int_{\Omega} \bar{v}_{sx} \cdot \nabla \bar{v}_{sx} \cdot \bar{v}_{sxx} dx + \int_{\Omega} \bar{f}_{sxx} \cdot \bar{v}_{sxx} dx. \end{aligned}$$

Using the fact that  $\bar{v}_s$  is divergence free we integrate by parts in the first two integrals on the r.h.s. of (3.17). We also integrate by parts in the third integral. Applying the Hölder and Young inequalities we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\bar{v}_{sxx}\|_{L_2}^2 + \nu \|\nabla \bar{v}_{sxx}\|_{L_2}^2 &\leq \varepsilon \|\nabla \bar{v}_{sxx}\|_{L_2}^2 \\ &\quad + c(1/\varepsilon) \left( \int_{\Omega} |\bar{v}_{sxx}|^2 |\bar{v}_s|^2 dx + \int_{\Omega} |\bar{v}_{sx}|^4 dx + \int_{\Omega} |\bar{f}_{sx}|^2 dx \right). \end{aligned}$$

Hence for sufficiently small  $\varepsilon$ , from inequality (2.5) we get

$$(3.18) \quad \frac{d}{dt} \|\bar{v}_{sxx}\|_{L_2}^2 + c_{s1} \|\bar{v}_{sxx}\|_{H^1}^2 \leq c(\|\bar{v}_s\|_{L^\infty}^2 \|\bar{v}_{sxx}\|_{L_2}^2 + \|\bar{v}_{sx}\|_{L_4}^4 + \|\bar{f}_{sx}\|_{L_2}^2).$$

Now, (3.18) implies that for  $t \in (kT, (k+1)T)$

$$(3.19) \quad \begin{aligned} &\frac{d}{dt} \left( \|\bar{v}_{sxx}\|_{L_2}^2 \exp \left( c_{s1}t - c_{s3} \int_{kT}^t \|\bar{v}_s\|_{H^2}^2 dt' \right) \right) \\ &\leq c_{s3} \|\bar{f}_{sx}\|_{L_2}^2 \exp \left( c_{s1}t - c_{s3} \int_{kT}^t \|\bar{v}_s\|_{H^2}^2 dt' \right), \end{aligned}$$

where we have used the estimates  $\|\bar{v}_{sx}\|_{L_4} \leq c\|\bar{v}_s\|_{H^2}$  and  $\|\bar{v}_{sx}\|_{L_4} \leq c\|\bar{v}_{sxx}\|_{L_2}$ .

Integrating (3.19) with respect to time from  $kT$  to  $t \in (kT, (k+1)T]$ ,  $k \in \mathbb{N}_0$ , yields

$$(3.20) \quad \begin{aligned} \|\bar{v}_{sxx}(t)\|_{L_2}^2 &\leq c_{s3} \exp\left(-c_{s1}t + c_{s3} \int_{kT}^t \|\bar{v}_s(t')\|_{H^2}^2 dt'\right) \\ &\cdot \int_{kT}^t \|\bar{f}_{sxx}(t')\|_{L_2}^2 \exp\left(c_{s1}t' - c_{s3} \int_{kT}^{t'} \|\bar{v}_s(t'')\|_{H^2}^2 dt''\right) dt' \\ &+ \|\bar{v}_{sxx}(kT)\|_{L_2}^2 \exp\left(-c_{s1}(t - kT) + c_{s3} \int_{kT}^t \|\bar{v}_s(t')\|_{H^2}^2 dt'\right). \end{aligned}$$

Using notation 1. and (3.6) we obtain from (3.20) the inequality

$$\begin{aligned} \|\bar{v}_{sxx}(t)\|_{L_2}^2 &\leq c_{s3} \exp\left(\frac{c_{s3}}{c_{s1}} A_8^2\right) A_{10}^2 \\ &+ \|\bar{v}_{sxx}(kT)\|_{L_2}^2 \exp\left(-c_{s1}(t - kT) + \frac{c_{s3}}{c_{s1}} A_8^2\right). \end{aligned}$$

In view of notation 2. we have

$$(3.21) \quad \|\bar{v}_{sxx}(t)\|_{L_2}^2 \leq A_{11}^2 + \|\bar{v}_{sxx}(kT)\|_{L_2}^2 \exp(-c_{s1}(t - kT) + c_{s4} A_8^2).$$

For  $t = (k+1)T$ , inequality (3.21) takes the form

$$\|\bar{v}_{sxx}((k+1)T)\|_{L_2}^2 \leq A_{11}^2 + \|\bar{v}_{sxx}(kT)\|_{L_2}^2 \exp(-c_{s1}T + c_{s4} A_8^2).$$

Assumption 3. implies

$$\|\bar{v}_{sxx}((k+1)T)\|_{L_2}^2 \leq A_{11}^2 + e^{-c_{s1}T/2} \|\bar{v}_{sxx}(kT)\|_{L_2}^2.$$

Hence, iteration yields

$$(3.22) \quad \begin{aligned} \|\bar{v}_{sxx}(kT)\|_{L_2}^2 &\leq \frac{A_{11}^2}{1 - e^{-c_{s1}T/2}} + e^{-c_{s1}kT/2} \|\bar{v}_{sxx}(0)\|_{L_2}^2 \\ &\leq 2A_{11}^2 + \|\bar{v}_{sxx}(0)\|_{L_2}^2 = A_{12}^2, \end{aligned}$$

where Assumption 4 is utilized. Employing (3.22) in (3.21) gives (3.15).

Integrating (3.18) with respect to time implies the estimate

$$\|\bar{v}_{sxx}(t)\|_{L_2}^2 + c_{s1} \int_{kT}^t \|\bar{v}_{sxx}(t')\|_{H^1}^2 dt' \leq c_{s3} A_{13}^2 A_8^2 + c_{s3} A_{10}^2 + A_{12}^2 = A_{14}^2$$

for all  $t \in (kT, (k+1)T]$ ,  $k \in \mathbb{N}_0$ . This implies (3.16) and concludes the proof.  $\square$

**Remark 3.5.** Applying Faedo-Galerkin approximations and using Lemmas 3.1, 3.2, 3.4 and estimates (3.15)–(3.16), we conclude that the assertion of Theorem 1.1 holds. Employing additionally Lemma 3.3 we obtain Theorem 1.2.

#### 4. Stability

To prove the stability of two-dimensional solutions we have to find solutions to problem (1.3) such that the inequality  $\|u(0)\|_{H^1} \leq \gamma$  implies that  $\|u(t)\|_{H^1} \leq c\gamma$  for  $\gamma$  sufficiently small and for all  $t \in \mathbb{R}_+$ , where  $c > 0$  is a constant.

First we derive an energy type estimate for solutions to problem (1.6).

**Lemma 4.1.** *Let the assumptions of Lemmas 3.1, 3.2 hold. Assume that  $g \in L_2(kT, (k+1)T; L_2(\Omega))$ ,  $k \in \mathbb{N}_0$  and  $\bar{u}$  satisfies (1.6). Assume that*

1.  $B_1^2 = \sup_k \int_{kT}^{(k+1)T} \|\bar{g}(t')\|_{L_2}^2 dt' < \infty$ ,
2.  $B_2^2 = \sup_k \int_{kT}^{(k+1)T} \left| \int_0^t \int_{\Omega} g dx dt' + \int_{\Omega} u(0) dx \right|^2 dt < \infty$ ,
3.  $B_3^2 = (c_2 B_1^2 + c_2 A_3^2 B_2^2) \exp(c_2 A_8^2)$ , where  $c_1 > 0$  is the constant from (2.6) and  $c_2 > 0$  appears in (4.3).
4.  $T$  is so large that  $-c_1 T/2 + A_8^2 \leq 0$ ,  $A_8^2$  appears in (3.13),
5.  $T$  is so large that  $1 - e^{-c_1 T/2} \geq 1/2$ ,
6.  $B_4^2 = B_3^2 + \exp(c_2 A_8^2)(2B_4^2 + \|\bar{u}(0)\|_{L_2}^2)$ .

Then

$$(4.1) \quad \|\bar{u}(t)\|_{L_2}^2 + c_1 \int_{kT}^t \|\bar{u}(t')\|_{H^1}^2 dt' \leq c_2 A_8^2 B_4^2 + c_2 A_3^2 B_2^2 + c_2 B_1^2 + B_3^2 \equiv B_5^2$$

for all  $t \in (kT, (k+1)T)$ ,  $k \in \mathbb{N}_0$ .

**Proof.** Multiplying (1.6)<sub>1</sub> by  $\bar{u}$ , integrating over  $\Omega$ , by parts and using the periodic boundary conditions we obtain

$$(4.2) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\bar{u}\|_{L_2}^2 + \nu \|\nabla \bar{u}\|_{L_2}^2 = - \int_{\Omega} u \cdot \nabla \bar{v}_s \cdot \bar{u} dx + \int_{\Omega} \bar{g} \cdot \bar{u} dx \\ & = - \int_{\Omega} (\bar{u} + \oint_{\Omega} u dx) \cdot \nabla \bar{v}_s \cdot \bar{u} dx + \int_{\Omega} \bar{g} \cdot \bar{u} dx \\ & = - \int_{\Omega} \bar{u} \cdot \nabla \bar{v}_s \cdot \bar{u} dx + \oint_{\Omega} u dx \cdot \int_{\Omega} \bar{v}_s \cdot \nabla \bar{u} dx + \int_{\Omega} \bar{g} \cdot \bar{u} dx. \end{aligned}$$

Using the Hölder and Young inequalities we get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\bar{u}\|_{L_2}^2 + \nu \|\nabla \bar{u}\|_{L_2}^2 \leq \varepsilon (\|\bar{u}\|_{L_6}^2 + \|\bar{u}_x\|_{L_2}^2 + \|\bar{u}\|_{L_2}^2) \\ & + c(\varepsilon) \left( \|\bar{v}_{sx}\|_{L_3}^2 \|\bar{u}\|_{L_2}^2 + \|\bar{v}_s\|_{L_2}^2 \left| \int_0^t \int_{\Omega} g dx dt' + \int_{\Omega} u(0) dx \right|^2 + \|\bar{g}\|_{L_2}^2 \right). \end{aligned}$$

Assuming that  $\varepsilon$  is sufficiently small and applying inequality (2.6) yields

$$(4.3) \quad \begin{aligned} & \frac{d}{dt} \|\bar{u}\|_{L_2}^2 + c_1 \|\bar{u}\|_{H^1}^2 \leq c_2 \|\bar{v}_{sx}\|_{L_3}^2 \|\bar{u}\|_{L_2}^2 \\ & + c_2 \|\bar{v}_s\|_{L_2}^2 \left| \int_0^t \iint_{\Omega} g dx dt' + \int_{\Omega} u(0) dx \right|^2 + c_2 \|\bar{g}\|_{L_2}^2, \end{aligned}$$

where  $c_1$  is the constant from (2.6). Inequality (4.3) implies

$$(4.4) \quad \begin{aligned} & \frac{d}{dt} \left( \|\bar{u}\|_{L_2}^2 e^{c_1 t - c_2 \int_{kT}^t \|\bar{v}_s\|_{H^2}^2 dt'} \right) \leq c_2 \left( \|\bar{g}\|_{L_2}^2 \right. \\ & \left. + \|\bar{v}_s\|_{L_2}^2 \left| \int_0^t \iint_{\Omega} g dx dt' + \int_{\Omega} u(0) dx \right|^2 \right) e^{c_1 t - c_2 \int_{kT}^t \|\bar{v}_s(t')\|_{H^2}^2 dt'} \end{aligned}$$

for all  $t \in (kT, (k+1)T]$ ,  $k \in \mathbb{N}_0$ .

Integrating (4.4) with respect to time from  $kT$  to  $t \in (kT, (k+1)T]$  yields

$$(4.5) \quad \begin{aligned} & \|\bar{u}(t)\|_{L_2}^2 \leq c_2 \exp \left( -c_1 t + c_2 \int_{kT}^t \|\bar{v}_s(t')\|_{H^2}^2 dt' \right) \\ & \cdot \int_{kT}^t \left( \|\bar{g}(t')\|_{L_2}^2 + A_3^2 \left| \int_0^{t'} \iint_{\Omega} g dx dt'' + \int_{\Omega} u(0) dx \right|^2 \right) e^{c_1 t'} dt' \\ & + \exp \left( -c_1(t - kT) + c_2 \int_{kT}^t \|\bar{v}_s(t')\|_{H^2}^2 dt' \right) \|\bar{u}(kT)\|_{L_2}^2, \end{aligned}$$

where (3.2) is used. In view of Assumptions 1–4 and (3.13) we have

$$\|\bar{u}(t)\|_{L_2}^2 \leq B_3^2 + e^{-c_1(t-kT) + c_2 A_8^2} \|\bar{u}(kT)\|_{L_2}^2.$$

Setting  $t = (k+1)T$  and using Assumption 5 we get

$$\|\bar{u}((k+1)T)\|_{L_2}^2 \leq B_3^2 + e^{-c_1 T/2} \|\bar{u}(kT)\|_{L_2}^2.$$

Hence, iteration implies

$$(4.6) \quad \|\bar{u}(kT)\|_{L_2}^2 \leq \frac{B_3^2}{1 - e^{-c_1 T/2}} + e^{-c_1 kT/2} \|\bar{u}(0)\|_{L_2}^2 \leq 2B_3^2 + \|\bar{u}(0)\|_{L_2}^2,$$

where Assumption 6 is used. Inserting (4.6) in (4.5) yields

$$\begin{aligned} \|\bar{u}(t)\|_{L_2}^2 & \leq B_3^2 + e^{-c_1(t-kT) + c_2 A_8^2} (2B_3^2 + \|\bar{u}(0)\|_{L_2}^2) \\ & \leq B_3^2 + e^{c_2 A_8^2} (2B_4^2 + \|\bar{u}(0)\|_{L_2}^2) \equiv B_4^2. \end{aligned}$$

Integrating (4.3) with respect to time from  $kT$  to  $t \in (kT, (k+1)T]$  we derive

$$\|\bar{u}(t)\|_{L_2}^2 + c_1 \int_{kT}^t \|\bar{u}(t')\|_{H^1}^2 dt' \leq c_2 A_8^2 B_4^2 + c_2 A_3^2 B_2^2 + c_2 B_1^2 + B_3^2.$$

This implies (4.1) and concludes the proof.  $\square$

Now, we show that the 3d solution to (1.1) remains close to the 2d solution of (1.2) if they are sufficiently close at the initial time.



**Lemma 4.2.** *Let the assumptions of Lemma 4.1 hold. Let  $\gamma_*$  be so small that  $c_1 - c_3\gamma_*^4 \geq c_1/2$ , where  $c_1 > 0$  is the constant from (2.6) and  $c_3 > 0$  occurs in (4.10)–(4.11). Let  $\gamma \in (0, \gamma_*]$ . Assume that*

$$\|\bar{u}(0)\|_{H^1}^2 \leq \gamma,$$

$$c_3\|v_{sx}\|_{L_3}^2 \left[ \|\bar{v}_{sx}(t)\|_{L_3}^2 B_5^2 + \left| \int_0^t \int_{\Omega} g dx dt + \int_{\Omega} u(0) dx \right|^2 \right] + c_3\|\bar{g}\|_{L_2}^2 \leq \frac{c_1}{4}\gamma \quad \text{for all } t \in \mathbb{R}_+.$$

Then

$$\|\bar{u}(t)\|_{H^1}^2 \leq \gamma \quad \text{for any } t \in \mathbb{R}_+.$$

This means that the 3d solution to (1.1) remains close to the 2d solution of (1.2) if their initial data and the external forces for all time are sufficiently close.

**Proof.** Differentiating (1.6)<sub>1</sub> with respect to  $x$ , multiplying the result by  $\bar{u}_x$ , integrating over  $\Omega$ , by parts and employing the periodic boundary conditions we obtain

$$(4.7) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \|\bar{u}_x\|_{L_2}^2 + \nu \|\bar{u}_{xx}\|_{L_2}^2 &\leq \|\bar{u}_x\|_{L_3}^3 + \left| \int_{\Omega} \bar{v}_{sx} \cdot \nabla u \cdot \bar{u}_x dx \right| \\ &+ 2 \left| \int_{\Omega} \bar{u}_x \cdot \nabla \bar{v}_s \cdot \bar{u}_x dx \right| + \left| \int_{\Omega} u \cdot \nabla \bar{v}_s \cdot \bar{u}_{xx} dx \right| + \left| \int_{\Omega} \bar{g} \cdot \bar{u}_{xx} dx \right|. \end{aligned}$$

Adding (4.2) and (4.7), applying the Hölder and Young inequalities, we derive

$$\begin{aligned} \frac{d}{dt} \|\bar{u}\|_{H^1}^2 + c_1 \|\bar{u}\|_{H^2}^2 &\leq c(\|\bar{u}_x\|_{L_3}^3 + \|\bar{v}_{sx}\|_{L_3}^2 \|\bar{u}_x\|_{L_2}^2 \\ &+ \|\bar{v}_{sx}\|_{L_3}^2 \|\bar{u}_x\|_{L_2}^2 + \|u\|_{L_6}^2 \|\bar{v}_{sx}\|_{L_3}^2 + \|\bar{g}\|_{L_2}^2). \end{aligned}$$

Using  $\|u\|_{L_6}^2 \leq \|\bar{u}\|_{L_6}^2 + \left| \int_{\Omega} u dx \right|^2$  and  $\|\bar{u}\|_{L_6} \leq c\|\bar{u}\|_{H^1} \leq c\|\bar{u}_x\|_{L_2}$ , which holds in view of the Poincaré inequality, we get

$$(4.8) \quad \frac{d}{dt} \|\bar{u}\|_{H^1}^2 + c_1 \|\bar{u}\|_{H^2}^2 \leq c \left[ \|\bar{u}_x\|_{L_3}^3 + \|\bar{v}_{sx}\|_{L_3}^2 \|\bar{u}_x\|_{L_2}^2 + \|v_{sx}\|_{L_3}^2 \left| \int_{\Omega} u dx \right|^2 + \|\bar{g}\|_{L_2}^2 \right].$$

In view of (2.2) and the interpolation inequality (see [4, Ch. 3, Sect. 15])

$$\|\bar{u}_x\|_{L_3} \leq c \|\bar{u}_{xx}\|_{L_2}^{1/2} \|\bar{u}_x\|_{L_2}^{1/2}$$

(which holds without the lower order term because  $\int_{\Omega} \bar{u}_x dx = 0$ ), we obtain from (4.8) the inequality

$$(4.9) \quad \begin{aligned} \frac{d}{dt} \|\bar{u}\|_{H^1}^2 + c_1 \|\bar{u}\|_{H^2}^2 &\leq c \|\bar{u}_x\|_{L_2}^6 + c \|\bar{v}_{sx}\|_{L_3}^2 \|\bar{u}_x\|_{L_2}^2 \\ &+ c \|v_{sx}\|_{L_3}^2 \left( \left| \int_0^t \int_{\Omega} g dx dt' + \int_{\Omega} u(0) dx \right|^2 \right) + c \|\bar{g}\|_{L_2}^2. \end{aligned}$$

Employing the interpolation inequality (see [4, Ch. 3, Sect. 10])

$$\|\bar{u}_x\|_{L_2} \leq \varepsilon^{1/2} \|\bar{u}_{xx}\|_{L_2} + c\varepsilon^{-1/2} \|\bar{u}\|_{L_2}$$

in (4.9) implies

$$(4.10) \quad \begin{aligned} & \frac{d}{dt} \|\bar{u}\|_{H^1}^2 + c_1 \|\bar{u}\|_{H^2}^2 \leq c_3 \|\bar{u}_x\|_{L_2}^6 \\ & + c_3 \|\bar{v}_{sx}\|_{L_3}^2 \|\bar{u}\|_{L_2}^2 + c_3 \|\bar{v}_{sx}\|_{L_3}^2 \left( \left| \int_0^t \int_{\Omega} g dx dt' + \int_{\Omega} u(0) dx \right|^2 \right) + c_3 \|\bar{g}\|_{L_2}^2. \end{aligned}$$

In view of (4.1) we have  $\|\bar{u}(t)\|_{L_2} \leq B_6$ . Hence we can introduce the quantities:

$$\begin{aligned} G^2(t) &= c_3 \|\bar{v}_{sx}(t)\|_{L_3}^2 \left[ B_6^2 + \left| \int_0^t \int_{\Omega} g dx dt' + \int_{\Omega} u(0) dx \right|^2 \right] + c_4 \|\bar{g}(t)\|_{L_2}^2, \\ X(t) &= \|\bar{u}(t)\|_{H^1}, \quad Y(t) = \|\bar{u}(t)\|_{H^2}. \end{aligned}$$

Then (4.10) takes the form

$$\frac{d}{dt} X^2 \leq -c_1 Y^2 + c_3 X^4 X^2 + G^2.$$

Since  $X \leq Y$  we have

$$(4.11) \quad \frac{d}{dt} X^2 \leq -X^2(c_1 - c_3 X^4) + G^2.$$

Let  $\gamma \in (0, \gamma_*]$ , where  $\gamma_*$  is so small that

$$(4.12) \quad c_1 - c_3 \gamma_*^4 \geq c_1/2.$$

By the assumptions of the lemma,

$$X^2(0) \leq \gamma, \quad G^2(t) \leq c_1 \frac{\gamma}{4} \quad \text{for all } t \in \mathbb{R}_+.$$

Suppose that

$$t_* = \inf\{t \in \mathbb{R}_+ : X^2(t) > \gamma\} > 0.$$

Then by (4.12) for  $t \in (0, t_*]$  inequality (4.11) takes the form

$$(4.13) \quad \frac{d}{dt} X^2 \leq -\frac{c_1}{2} X^2 + G^2(t).$$

Clearly, we have

$$(4.14) \quad X^2(t_*) = \gamma \quad \text{and} \quad X^2(t) > \gamma \quad \text{for } t > t_*.$$

Then (4.13) yields

$$\left. \frac{d}{dt} X^2(t) \right|_{t=t_*} \leq c_1 \left( -\frac{\gamma}{2} + \frac{\gamma}{4} \right) < 0$$

contradicting with (4.14). Therefore

$$(4.15) \quad X^2(t) < \gamma \quad \text{for } t \in \mathbb{R}_+.$$

This concludes the proof.  $\square$

**Lemma 4.3.** *Let  $A_8$  be as introduced in (3.13),  $A_9$  as in Lemma 3.3 and  $\gamma$  as in Lemma 4.2. Let  $T$  be as defined in Lemma 4.1. Let*

$$B_6 = \sup_k \sup_{kT \leq t \leq (k+1)T} \left| \int_0^t \int_{\Omega} g(t) dx dt + \int_{\Omega} u(0) dx \right|.$$

*Then there exists a solution to problem (1.6) such that  $\bar{u} \in H^{2,1}(\Omega \times (kT, (k+1)T))$ ,  $\nabla \bar{q} \in L_2(kT, (k+1)T; L_2(\Omega))$ ,  $k \in \mathbb{N}_0$ , and*

$$\begin{aligned} & \|\bar{u}\|_{H^{2,1}(\Omega \times (kT, (k+1)T))}^2 + \|\nabla \bar{q}\|_{L_2(kT, (k+1)T; L_2(\Omega))}^2 \\ & \leq c[(T+1)\gamma^2 + B_6^2][A_8^2(1 + A_8^2 + A_9^2) + A_9^2 + (T+1)\gamma^2] + c\gamma^2 \equiv B_7^2. \end{aligned}$$

**Proof.** In view of the definition of  $G$  we express (4.10) in the form

$$(4.16) \quad \frac{d}{dt} \|\bar{u}\|_{H^1}^2 + c_1 \|\bar{u}\|_{H^2}^2 \leq c_3 \|\bar{u}_x\|_{L_2}^6 + G^2.$$

Integrating (4.16) with respect to time from  $kT$  to  $t \in [kT, (k+1)T]$  and using (4.15) we derive

$$(4.17) \quad \|\bar{u}(t)\|_{H^1}^2 + c_1 \int_{kT}^t \|\bar{u}(t')\|_{H^2}^2 dt' \leq c_3 \gamma^6 T + \gamma^2 T + \gamma^2 \leq c(T+1)\gamma^2,$$

because  $\gamma < 1$ . Multiplying (1.6) by  $\bar{u}_t$  and integrating over  $\Omega \times (kT, (k+1)T)$  yields

$$\begin{aligned} (4.18) \quad & \int_{kT}^{(k+1)T} \|\bar{u}_t(t)\|_{L_2}^2 dt + \nu \|\nabla \bar{u}((k+1)T)\|_{L_2}^2 \\ & \leq c \int_{kT}^{(k+1)T} (\|u \cdot \nabla \bar{u}\|_{L_2}^2 + \|v_s \cdot \nabla \bar{u}\|_{L_2}^2 + \|u \cdot \nabla \bar{v}_s\|_{L_2}^2) dt \\ & \quad + c \int_{kT}^{(k+1)T} \|\bar{g}(t)\|_{L_2}^2 dt + \nu \|\nabla \bar{u}(kT)\|_{L_2}^2. \end{aligned}$$

The first term on the r.h.s. of (4.18) is estimated by

$$\begin{aligned} & c \sup_{kT \leq t \leq (k+1)T} \|u(t)\|_{H^1}^2 (\|\bar{u}\|_{L_2(kT, (k+1)T; H^2)}^2 + \|\bar{v}_s\|_{L_2(kT, (k+1)T; H^2)}^2) \\ & \quad + c \sup_{kT \leq t \leq (k+1)T} \|v_s(t)\|_{H^1}^2 \|\bar{u}\|_{L_2(kT, (k+1)T; H^2)}^2 \\ & \leq c(\gamma^2 + B_6^2)[A_8^2(1 + A_8^2) + A_8^2 A_9^2 + (T+1)\gamma^2] + c(A_8^2 + A_9^2)(T+1)\gamma^2 \\ & \leq B_7^2. \end{aligned}$$

Similarly,

$$\begin{aligned} & \|\nabla \bar{q}\|_{L_2(kT, (k+1)T; L_2)} \leq c \|\bar{u}\|_{H^{2,1}(\Omega \times (kT, (k+1)T))} \\ & \quad + c \|u \cdot \nabla \bar{u}\|_{L_2(kT, (k+1)T; L_2)} + c \|v_s \cdot \nabla \bar{u}\|_{L_2(kT, (k+1)T; L_2)} \\ & \quad + c \|u \cdot \nabla \bar{v}_s\|_{L_2(kT, (k+1)T; L_2)} + \|\bar{g}\|_{L_2(kT, (k+1)T; L_2)} \leq cB_7. \end{aligned}$$

This concludes the proof.  $\square$

Now, we can complete the proofs of Theorems 1.3 and 1.4.

### The proofs of Theorems 1.3 and 1.4

Inequalities (1.7) and (1.9) follow from Lemmas 4.2 and 4.3, respectively. The existence of solution is a consequence of applying the Faedo-Galerkin method and inequalities (1.7)–(1.8). Thus, we get the assertion of Theorem 1.3. Theorem 1.4 follows from Lemma 4.3.  $\square$

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